My Objective in this Conversational Overview

- Is to raise an awareness to the following matters, namely, that:

- An appreciation of the underpinnings of the ***science of quantifying uncertainty*** is germane for probabilists, statisticians, and data scientists.

- The subjectivity of probability, espoused by de Finetti and Ramsey, was inspired by the *positivist* ideas of Mach and Einstein, that ascribe meaning only to observable, measurable, and actionable, entities.

- To de Finetti and Ramsey uncertainty is measured by how you bet on uncertain events or how you make decisions under uncertainty.

- Subjectivity of probability is now at the very doorstep of *quantum theory*, the most powerful theory in physics, if not all of the sciences.

- Pure probability ***untangled*** from personal preference cannot exist!

Two Notable Quotes for a Start

- *" …, all sciences would only be an unconscious application of the calculus of probabilities. And if this calculus be condemned, then the whole of sciences must also be condemned"*.

  Poincare (1905) – his emphasis is on "calculus".
- Poincare was alluding to the Bayes/La Place "***probabilistic induction***"; i.e. the probability of the cause of an event, versus the then prevailing Newtonian dictum on the certainty of a cause.

- ***Quotes from a 1944 landmark mathematics publication***:

- "The practical person asks, what is the meaning of a probability statement?

- How should I act on it?

- If the mathematician succeeds in answering this question, the philosopher wants to know the reason for the answer.

- What we wish to emphasize is that the mathematician has not, and cannot, answer questions such as "how is the probability of concretely given events defined?"

  – Paul Halmos (1944). *American Mathematics Monthly*.

Roadmap of this Talk

0. The Origins of Probability and of Empirical Statistics.
1. The Several Interpretations of Probability.
2. Critique of the Frequency, Classical, and Logical Interpretations.
3. Subjective Probability and its Operationalization.
4. A Closed Book Non-Technical Quiz (for faculty only).
5. The Architectural Evolution of Subjective Probability.
6. Axioms, Paradoxes, and Acrobatics.

- ***Everything here has been a struggle existing writings*.**

Why This Talk at This Meeting?

- *Conclusions based on statistical inferences, and the data sciences are grounded in probabilistic concepts (independence, consistent estimators, confidence limits, p-values, objective priors), and difficulties with appreciating these notions stem from issues with interpreting probability.*
- *Probability theory owes its birth to the Pascal-Fermat correspondence of 1654 on problems of gambling. However, the word probability was never mentioned therein.*

Historical Roadmap

- *The Pascal-Fermat letters stimulated further research on gambling by Jacob Bernoulli (1655-1705) who introduced the term probability, based on the Latin "**probabilitas**" (a moral system of the Catholic church laid out by the Spanish Jesuit – Bartholome dé Medina), in Bernoulli's famous treatise "**Ars Conjectandi**" (or the **Science of Prediction**).*
- *Bernoulli's aim was to introduce a new science which he called "**stochastics**" (which in Greek means the Science of Prediction).*

Roadmap – Contd.

- *Bernoulli's title of Ars Conjectandi is a possible paraphrasing of "**Ars Cognitandi"** – The Art of Thinking - (which is a part of the four part treatise called **Port Royal Logic**.)*
- *To Bernoulli a key feature of stochastics is an event's readiness to occur, and the probability of an event ≡ degree of certainty of its occurrence.*
- *Thus Ars Conjectandi is the art of measuring probabilities (and this is what we all do).*

Bernoulli's Law of Large Numbers

- *Bernoulli recognized the difficulty of determining the true value of probability and labeled as "mad" any attempt to do so.*
- *This motivated his weak law of large numbers as an empirical method to determine a lower and upper limit for an unknown probability. This law presumes the existence of a probability; it does not make a case for probability!*
- *de Moivre's Doctrine of Chances (1718) was about calculating the probability of events, without addressing the matter of* ***what does probability mean?***

Kolmogorov's – Slick Evasion

- *Kolmogorov's axiomatic foundation was to establish a new branch of mathematics, wherein a theory of probability pertains to a system of sets which satisfy certain conditions. He used the term "probability" detached from any real world meaning.*

The Historical Landscape from Bernoulli to Kolmogorov

- *To summarize, Pascal-Fermat never mentioned the word probability, deMoivre took it for granted, to Bernoulli it was the **readiness of an event to occur** and difficulties measuring it, and to Kolmogorov it was an avoidance of meaning and an irrelevance to real world applications.*

Is There a Roadmap to Statistics?

- *Simultaneous to Bernoulli's work, data sets were compiled by John Graunt (1662) on demography by Petty (1690), on what he called **Political Arithmetic**, and by de Witt (1671) on actuarial mathematics.*
- *But most noteworthy is John Sinclair's 21 volume work, between 1791-1799, on "**Statistical Account of Scotland**", introducing for the first time the word **statistics** as a replacement to Petty's term Political Arithmetic.*

Does the Word Statistics Have a Meaning?

- *Sinclair's motivation for using the German word "Statistics" was to attract public attention. The purpose of his work, was to assess the strength of a country via its inhabitant's happiness.*

- *Thus in contrast to Bernoulli's stochastics, de Moivre's chances, and Petty's political arithmetic which have evidential meaning, **statistics is an artificial word** which has now come to stand for anything dealing with data.*

Interpretations of Probability

- The word "*probability*" and its synonyms, like "*likelihood*", and "*chance*" appear in all of the sciences, mathematics, and philosophy.

- Because of its use in a variety of contexts, these words have acquired several meanings, not clearly distinguishable from each other.

- When used in science and engineering, the term probability should have a definite meaning.

- Facetiously, to some “young turks”, there is no problem. Probability is a non-negative, additive, set function, whose maximum value is one!

- But how can this help explain how probability can be used? How can data scientists and statisticians communicate their conclusions?

- What did Kolmogorov have to informally say about the zenith of his achievement?

- His answer was a surprise and an enlightenment to me!

- What is the connection between this abstract mathematical entity and the pragmatic contexts of physics, biology, and engineering?

- Three types of connectives seemed to have appeared:
  - ***Empirical*** or ***Frequentist*** [due to Venn (1886), pursued by von Mises, and by Reichenbach], and all those engaged with consistent estimators and confidence limits.
  - ***Logical*** [due to Keynes (1921), pursued by Carnap, Richard Cox, and Harold Jeffreys].
  - ***Subjective*** or ***Personalistic*** [explicit in de Morgan (1847), implicit in Bernoulli (1713), Bayes (1763), and Laplace (1825)], and a handful of Bayesians.
- To philosophers Carnap and Nagel, probability admits both an empirical and a logical interpretation, and to the mathematician, **B. O. *Koopman***, a logical and a subjectivist interpretation.

- In the empirical interpretation, probability (or chance) is the limit of a relative frequency, and is a *property of a "kollektive",* not of an isolated single event.

- Here Probability is unique, reproducible, and physical; it is labeled ***objective*** *(even though, like magnetism, there does not exist a scientific experiment to prove its existence)*.

- An extreme alternative to the empirical view is the ***logical*** (or ***necessarist)*** view.

- This view denies that probability statements are empirical. Here probability is a logical, and unique, relationship between a proposition and a body of knowledge.

- Thus if knowledge says that a coin is perfectly balanced, then its probability of heads must be 1/2.

- In the subjectivist view, probability is also a relation between proposition $\mathcal{A}$ and evidence $\mathcal{H}$, but the relationship need not be purely logical.

- It is a *quasi-logical* relationship wherein probability p, is one's degree of belief in $\mathcal{A}$.

- Probability is a corporate state of mind governed by a system of axioms. Thus the claim "quasi-logical".

- However p need not be unique to all individuals.

- Thus in the necessarist and the subjectivist views, probability is interpreted as a ***degree of belief***.

- The subjectivist quasi- logical theory allows only certain combinations of degrees of belief in statements.

- The above idea, due to Ramsey (1926), a co-originator of the subjectivist theory, with de Finetti (1928), is called ***coherence***.

- Thus irrespective of interpretation, necessarist or subjectivist, a collection of ***degrees of belief should cohere***.

- Coherence means a strict adherence to three "*Axioms*": convexity, addition, <u>and</u> the multiplication rule.

- ***Note:*** In Kolmogorov's system, the multiplication rule is not an axiom; it is a <u>definition</u> (the ratio of two probabilities).

- Entering into this mix are two other notions: ***Intuitive Probability*** and ***Propensity***.

- The intuitive thesis in probability holds that probability derives directly from intuition, and is prior to objective experience (data).

- It holds that experience is to be interpreted in terms of probability and not the reverse [Koopman (1939) in *Annals of Mathematics*].

The Propensity Notion of Probability

- Proposed by the American philosopher Peirce.
- Developed by Popper (1957) to provide an interpretation of quantum theory, different from the Heisenberg-Bohr subjectivist view.
- Probability is a propensity (or a ***disposition*, or *tendency*)** of a physical situation to yield a certain outcome, or to yield a long run relative frequency of such an outcome.

- Popper claimed that propensities or ***chances*** are unobservable dispositional properties of the physical world, independent of our theorizing, and comparable to a Newtonian force.

- The idea of propensities is metaphysical in exactly the same sense as forces or fields of forces. That is, it can make meaningful assertions which cannot be falsified by observation alone.

- Propensities depend on the generating conditions of an outcome, and can be invoked for singular events.

Critique of Frequency Theory

- Consider the statement "the probability that a particular coin will fall heads when tossed is 1/2 ".

- According to the frequency interpretation the following is the most common:

  – "If the coin is tossed a <u>large</u> number of times under <u>similar</u> conditions it will fall heads in <u>approximately</u> half the tosses".

- This interpretation has many vague terms.

  i. How large is large? An infinite is metaphysical.

  ii. What does similar conditions mean? If the same conditions hold then you will get the same outcome. How different from same is similar?

  iii. How close to 1/2 does the term approximate mean?

- Notwithstanding the above, this interpretation applies only to (conceptually) repeatable experiments like coin tosses.

- ***Striking statements about frequency and probability.***

  i. "The frequency concept based on the notion of limiting frequency as the number of trials increases to ***infinity***, does not contribute anything to substantiate the applicability of the results of probability theory to real practical problems where the number of trials is always finite. The frequency concept applied to a large but ***finite*** number of trials does not admit a rigorous exposition within the framework of pure mathematics."

  ii. In discussing the law of large numbers about the closeness of x/n (the frequency) to p (the probability), this writer also says "will never allow us to be free of the necessity of referring to probabilities in the ***primitive imprecise*** sense of the term."

  – Kolmogorov (1963) and (1969), respectively.

Quote by a "Pukka" Bayesian

- Probability is a concept that makes sense only when it is regarded as a ***primitive***, something that is understood without definition, and revealed by verbal reports. A decision maker's probabilities are whatever he/she says.
- Morris De Groot (1970).

Critique of Classical Probability

- The notion most used by Bernoulli, Bayes, de Moivre, and Laplace, and in introductory texts, is based on the ***principle of sufficient reason***.

- We say the probability of a coin landing heads is ½ because there are only two possibilities, heads and tails, and they are equally likely. Thus heads and tails should be assigned the same probability.

- For example, d'Alembert's argument that if he tossed a coin two times there were three possibilities: 2 heads, 1 head, or 0 heads and each has probability ${}^{1}/_{3}$.

- Difficulties with the approach are:
   i. The definition is circular because the term 'likely' is used to define 'probability', and 'probability' to define 'likely'.

   ii. What if the coin is unbalanced so that we cannot regard heads and tails as equally likely?

   iii. It is physically impossible to produce a balanced coin, so the notion of balance is a subjective judgment anyway.

Choice Based Subjective Probability

- "We are driven therefore to the supposition that the degree of a belief . . ., which we can express (vaguely) as the extent to which we are prepared to act on it". [Ramsey (1931), p. 170].

- "The degree of probability exists only subjectively in the minds of individuals, and the probability attributed by an individual to a given event is revealed by the conditions under which he would be disposed to bet on the event". [de Finetti (1937)].

- "Personalistic views hold that probability measures the confidence that a particular individual has in the truth of a particular proposition". [Savage (1954)].

The Appearance of Utility with Subjective Probability

- The notions of utility, and maximization of expected utility were <u>conceived</u> in 1738 by N. Bernoulli for resolving the St. Petersburg Paradox; he assumed the existence of objective probabilities.
- Initially, in 1926, Ramsey took utilities as a given and sketched a proof for the <u>existence</u> of subjective probabilities, assuming that individuals make choices under uncertainty that maximize expected utilities.
- Ramsey's 1926 work paralleled that of de Finetti (1928), who operationalized subjective probability, by a linear utility, and no arbitrage, via his notion of a 2-sided bet.

Ramsey's Celebrated Proposal

- Later on, recognizing that an analysis predicated on a known system of numerical utilities suffers from defects like ones aversion to betting and the diminishing marginal utilities, Ramsey sketches a proposal for the ***simultaneous axiomatization*** of probability and utility based on a transitive system of preferences among various options.
- In so doing Ramsey in 1931, paved a path for the behavioristic foundations of subjective probability, a path formalized by Savage in 1954.

Entanglement of Subjective Probability and Utility

- von Neumann and Morgenstern in 1937 gave an axiomatic <u>characterization</u> of expected utility maximizers who may employ a randomizing device to determine the choice of a pure strategy assuming frequency based probabilities.
- Savage (1954) synthesized the ideas of Ramsey, de Finetti, and von Neumann- Morgenstern to introduce a new analytical framework and necessary and sufficient conditions for the <u>simultaneous</u> <u>existence</u> of a unique utility and a subjective probability and also a characterization of an expected utility maximizer.

Other Approaches to Inducing Subjective Probability

- Anscombe and Aumann (1962) take utility for granted and develop subjective probability by assuming the existence of an objective chance mechanism. In so doing, they link subjective probability and propensity to make the claim that their approach defines more difficult (subjective) probabilities in terms of the easier to digest objective probabilities.

- Pratt, Raiffa and Schlaifer (1964) also assume a chance mechanism, and derive the simultaneous existence of utility and subjective probability from first principles.

Betting Rates, Consequences, and State Dependence

- Ramsey (1926) and de Finetti (1928) developed their theories independently and contemporaneously.
- Savage (1954) synthesized their work and incorporated features of the von Neumann and Morgenstern's (1944) ***expected utility theory***.
- de Finetti's betting rates are for <u>money</u> and could depend on beliefs as well as a state dependent utility for money. They are therefore tangled.
- To Ramsey and Savage, betting rates are in terms of <u>consequences</u> having a state-independent utility. They wanted to disentangle belief from value.

Essence of Ramsey-de Finetti Quotes

- All propose the same behavioristic definition of probability, namely, the rate at which an individual is willing to bet on an event.
- The betting rates are primitive measurements that reveal your probability. They can be <u>informed</u> by classical, logical, or frequency based reasoning, or other reasons.
- In de Finetti's theory, probability is the price you are willing to pay for a lottery ticket which yields one unit of money if the event occurs and nothing otherwise.

Operationalizing Personal Probability

- de Finetti proposed a <u>two-sided</u> bet to operationalize personal probability.

- If $\mathcal{W}$ declares p as the probability of rain tomorrow to boss $\mathcal{C}$, then $\mathcal{W}$ is prepared to stake:

  i. p cents in exchange of $1 if it rains tomorrow, and also to stake

  ii. (1- p) cents in exchange of $1 if it <u>fails</u> to rain tomorrow.

- $\mathcal{C}$ gets to choose between *i*. or *ii*. – the side of the bet.

- This forces $\mathcal{W}$ to declare her honest probability so that $\mathcal{C}$'s ignorance about the weather is not exploited by $\mathcal{W}$.

Pure Probability Does Not Exist.

- Just like how it is impossible to measure position and momentum, with physical instruments (Heisenberg), it is impossible to measure degrees of pure belief with the instruments proposed by Savage and others.

- Belief and value are as inseparable as space and time (cf. Herman Rubin).

- Pure probability (uncontaminated by value) is therefore a mental construct that makes sense only when it is regarded as a primitive [that is something is understood without definition, and revealed by verbal reports (de Groot, 1970)].

- I therefore liken pure probability to Popper's propensity or to Kolmogorov's primitive, as a ***useful mental construct*** which should not bear the requirement of external validity.

- ***What must we then make of the answers we produce?***

Linking Pure and Subjective Probability

- The word "chance" appears in two metaphysical senses: Popper's propensity (or pure probability), and von Mises' limit of relative frequency, both seen as being objective.
- DeFinetti, acknowledges von Mises' chance as a useful entity, and by his ***theorem on exchangeability***, links it with subjective probability, which to de Finetti is ones disposition towards a two-sided bet.
- Anscombe and Aumann, do a similar link but via the von Neumann-Morgenstern axioms of utility theory and via the introduction of a canonical experiment having equally likely outcomes.

De Finetti's "Great" Theorem: A Foundation for Bayesian Inference

- de Finetti, under his extreme positivist Machian disposition, started with exchangeability (of observables), and showed that the joint distribution of binary random variables can be deduced in exactly the same manner as believing in a model of Bernoulli trials governed by an unknown chance p, with an uncertainty about p.
- Furthermore, the limiting frequency of the binary variables exists with a subjective probability one, and the limit is indeed the chance parameter p.

Statement of de Finetti's Theorem

- For an infinite binary exchangeable sequence $X_1, X_2, . .$

  P$(X_1 = x_1, \ldots X_n = x_n)$

  = $\int_0^1 p^{\sum x_i} (1 - p)^{n - \sum x_i} \mathbb{III}(p) dp,$

  where p = $\lim_{n \to \infty} \frac{\sum x_i}{n}$, $exists, and$

- $\mathbb{III}$(p) is a subjective (prior) probability density on an objective chance p.

Characterization of Statisticians

- The interpretation of probability is a key driver of statistical methodology, and the development of algorithms for information processing.

- The concept of probability almost unanimously adopted by statisticians is frequency. They are therefore called ***frequentists***; they have indeed produced a powerful arsenal of practical tools.

- By contrast, ***Bayesians*** are those statisticians who adopt the subjectivist concept of probability and use Bayes' Theorem (and the ***likelihood principle***) to develop their tools.

- ***Objective Bayesians*** – currently in the fastest growing segment of statisticians – employ the tools of the (subjective) Bayesians, but who uphold the thesis that probabilities need not be subjective, nor logical, nor unique to individuals.

- In so doing they may violate the requirement of coherence, namely, the Kolmogorov axioms.

What is The Theory of Subjective Probability?

- A theory which attempts to make precise the connection between ones coherent dispositions towards uncertainty, and mathematical probability as axiomatized by Kolmogorov (1933).
- It accommodates the Bayes-Laplace classical interpretation, the intuitive views of Koopman and Keynes, and the decision oriented approach of Ramsey, de Finetti, and Savage [Fishburn (1986)].
- In the development of this theory certain individuals have played founding roles.

My quiz is to match the names with the key ideas.

Closed Book Quiz for Faculty Only

Match the following notions with their originators.

Notion/Idea	*Originator*
Borel Sets	von Neumann, John
Countable Additivity in Probability	Lebesgue, Henri
Game Theory	Laplace, Pierre
Probability as a Disposition to Taking a Bet	Kolmogorov, Andrei
Measure Theory	de Finetti, Bruno
Professional Politician	Borel, Émile

Notion/Idea	Originator
Borel Sets	Borel
Countable Additivity in Probability	Borel
Game Theory	Borel
Probability as a Disposition to Taking a Bet	Borel
Measure Theory	Borel
Professional Politician	Borel

Subjective Probability: An Overview

- Each person needs to assign numerical probabilities to sub-sets of a sample space S, to reflect his/her beliefs.

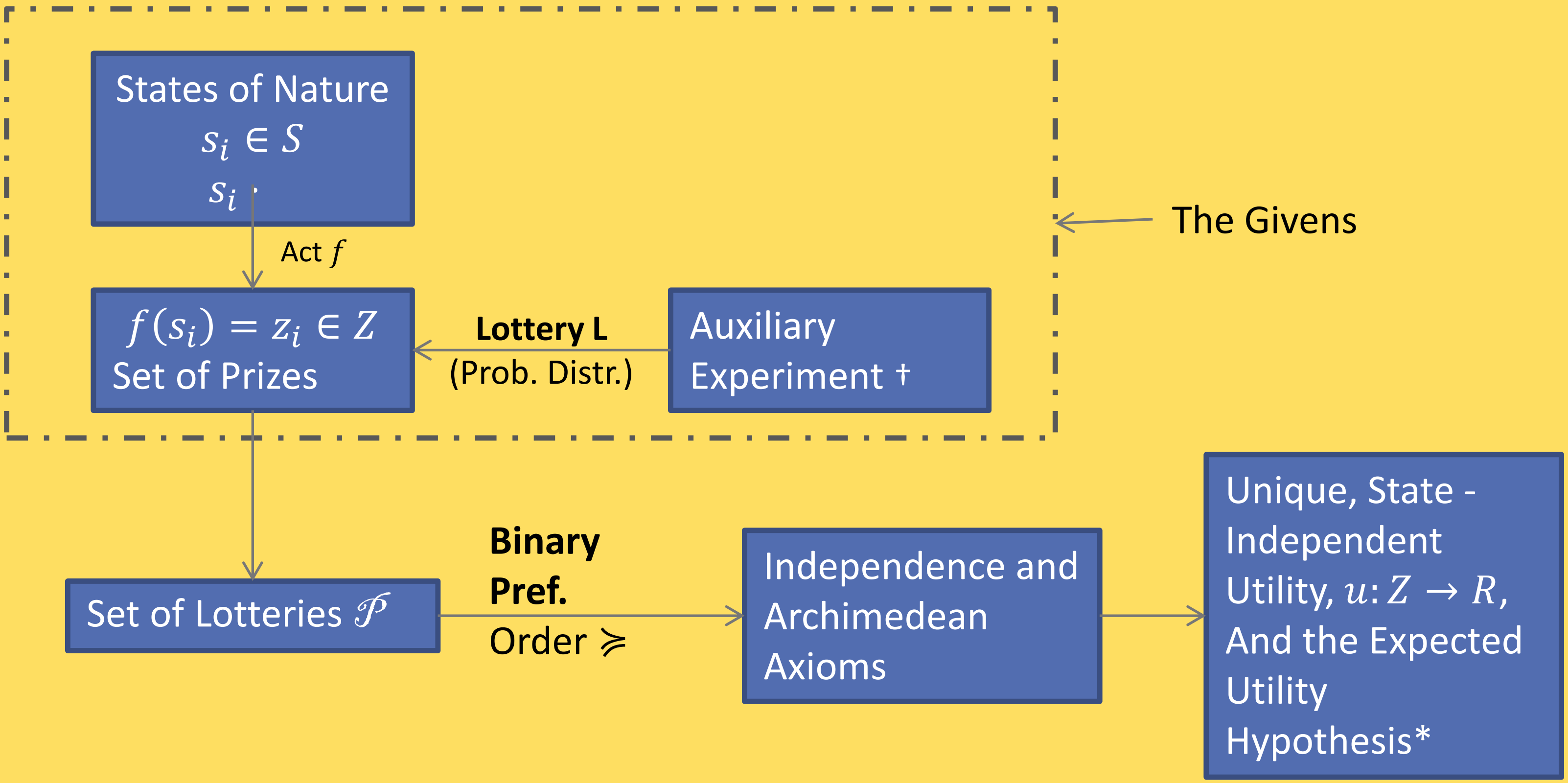
- Like Euclidian geometry, subjective probability begins with a list of undefined postulates, called ***axioms***, derived only after an examination of our intuitive notions of the subject.

- What axioms are needed to produce such probabilities from more basic qualitative relationships representing the person's judgments about:
- <u>preferences</u> between acts, and about
- events which are <u>more</u> <u>likely</u> to occur than the others?

Deducing Subjective Probability

- The existence of subjective probabilities is a consequence of the *principle of expected utility*, which is grounded in several axiomatic systems (Savage, Anscombe and Auman, Pratt et al., and deGroot).
- The expected utility hypothesis started with Daniel Bernoulli and the St. Petersburg paradox, but its mathematical foundation only came about almost 200 years later under von Neumann and Morgenstern's development of *Game Theory*.
- Central to all the axiom systems is the v N-M set up which can be seen as a true foundation for the formal development of subjective probability and utility.

von Neumann - Morgenstern Architecture



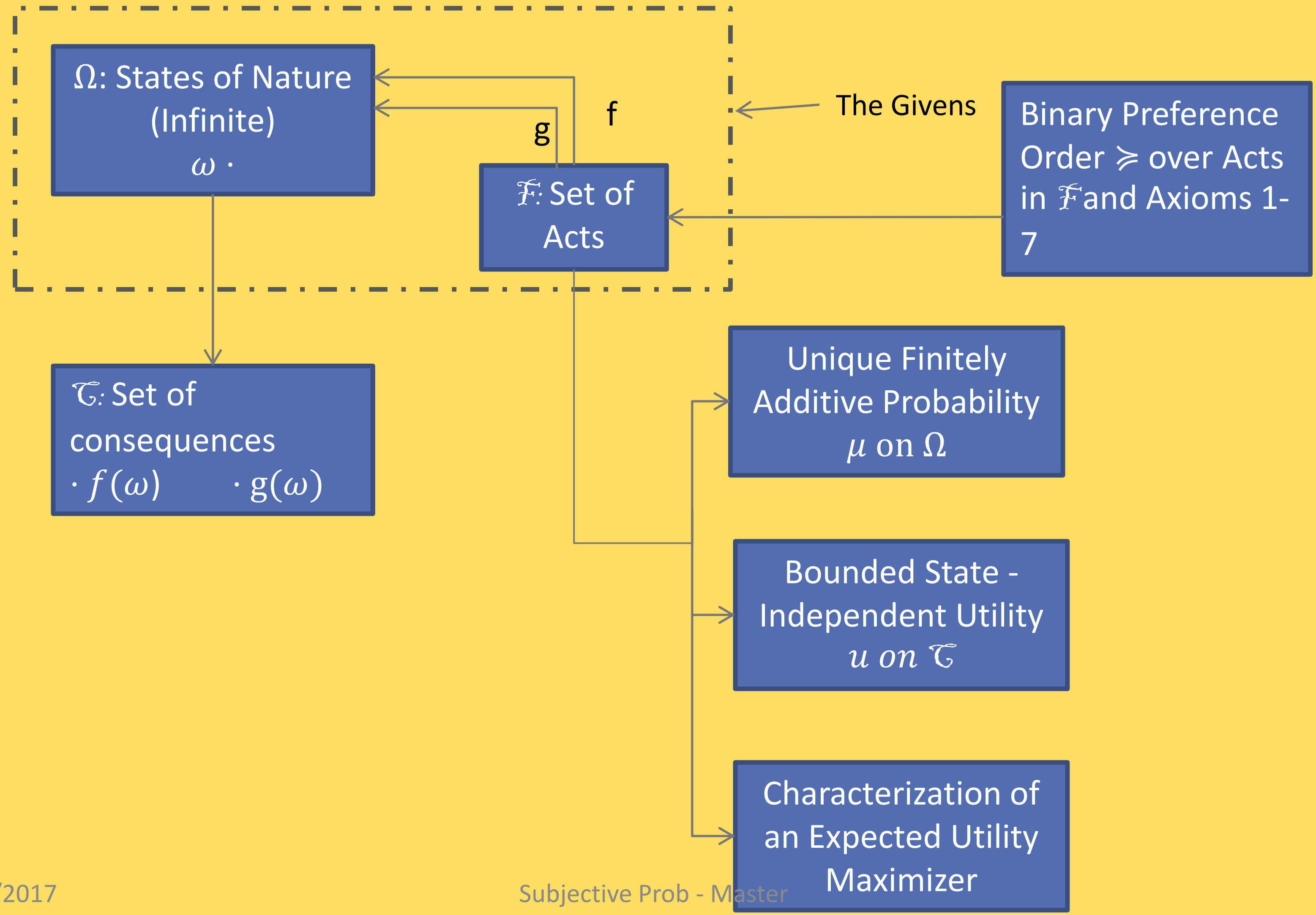
* $L_1, L_2 \in \mathscr{P}$ and $L_1 \succcurlyeq L_2 \iff \int_Z u(z)dL_1(z) \geq \int_Z u(z)dL_2(z)$

† An Objective Probability (Lottery) Generating Mechanism.

Concerns About v N-M's Architecture

- Assumes an objective probability distribution (or lottery), external to the system.
- The probability distributions are generated by an auxiliary physical experiment.
- The derived utility is *state independent*.
- Next is a pictorial display of Savage's work.

Savage's Probability and Utility



Concerns About Savage's Architecture

- For uniqueness of the induced subjective probability, the states of nature are infinite.
- The induced probability is only *finitely additive, and* the induced utility *state independent*.
- Anscombe and Aumann introduce ***horse lotteries*** (a map from state of nature to a lottery) and assuming a state dependent utility, together with the existence of a uniformly distributed random variable, induce a subjective probability.

De Groot's Approach

- Since de Finetti's betting approach is unable to disentangle probability from utility, that Savage's leads to state independence and countable additivity, and Anscombe and Aumann assume a utility, De Groot develops an approach deducing a unique countably additive probability (without utility) assuming an undefined primitive $\succcurlyeq$ among events, and the existence of a uniformly [0,1] distributed random quantity.
- The essence of De Groot's approach follows.

- For $A, B \in S$, there exists a binary relation $\succcurlyeq$ such that $A \succcurlyeq B$ or $B \succcurlyeq A$, or both.

"$A \succcurlyeq B$" $\Leftrightarrow$ A is regarded (by the person) to be at least as *likely to occur* as B

- If $A \succcurlyeq B$ and $B \succcurlyeq A \Leftrightarrow$ A and B are equally likely, written $A \sim B$.

- If $A \succcurlyeq B$ but $B \not\succcurlyeq A \Rightarrow A$ is more likely than B, written $A \succ B$.

- Important: $\succcurlyeq$ is an undefined primitive relationship and we assume that a person can make the judgment $A \succcurlyeq B$, or not, based on intuition alone.

- Definition: We say that a probability distribution P defined on the sub-sets of S ***agrees with*** $\succcurlyeq$ iff

$$A \succcurlyeq B \Leftrightarrow P(A) \geq P(B).$$

- Question: What axioms must be imposed on $\succcurlyeq$ to guarantee the existence of a unique P that agrees with it?

- ***Axiom 1***: For any $A, B \in S$, exactly one of the following hold:

$$A \succ B \text{ or } B \succ A \text{ or } A \sim B\ .$$

- ***Axiom 2***: If A_1, A_2, B_1, and $B_2 \in S$ are such that $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$ and $A_1 \succcurlyeq B_1$ and $A_2 \succcurlyeq B_2$ then $A_1 \cup A_2 \succcurlyeq B_1 \cup B_2$.

  Furthermore if $A_1 \succ B_1$ then $A_1 \cup A_2 \succ B_1 \cup B_2$.

- **Theorem 1**: Axioms 1 and 2 $\Rightarrow$ $\succcurlyeq$ is transitive. That is, if $A_1 \succcurlyeq A_2$ and $A_2 \succcurlyeq A_3 \Rightarrow A_1 \succcurlyeq A_3$, and if either $A_1 \succ A_2$ or $A_2 \succ A_3$ then $A_1 \succ A_3$.

- **Result**: Theorem 1 $\Rightarrow$ the relationship $\succcurlyeq$ yields a total ordering of all the subsets of S.

- ***Axiom 3***: $S \succ \emptyset$ and $A \succcurlyeq \emptyset$ for every $A \in S$.

  This states that the certain event S is more likely than the impossible event $\emptyset$, and no event is less likely than $\emptyset$.

- **Theorem 2**: Axioms 1 through 3 imply that if $A \subset B$, then $B \succcurlyeq A$.

- ***Axiom 4***: This is a technical condition pertaining to the continuity of the relationship $\succcurlyeq$.

If $A_1 \supset A_2 \supset A_3$ ...and $A_n \succcurlyeq B$ for $n = 1,2, \ldots$, then

$$\bigcap_{1}^{\infty} A_n \succcurlyeq B.$$

- Axioms 1-3 have their origins in the work of Bernstein (1917), improvised by de Finetti (1931), and by Koopman (1940).

- de Finetti & Koopman derive an additive probability measure from a qualitative probability under the assumption (an additional axiom) that for every integer n, there are n mutually exclusive equally probable events.

- Axiom 4 is due to Villegas (1964).

- In a classic paper, Kraft, Pratt and Seidenberg (1959) show that there are qualitative probabilities on finite Boolean algebras that do not agree with $\succcurlyeq$.

- Thus Axioms 1-4 are not sufficient to guarantee the existence of a unique P. Therefore Axiom 5.

- ***Axiom 5 (The Axiom of Acrobatics)***:

  There exists a random quantity X with a ***uniform*** distribution on $[0, 1]$.

- ***Definition of a Uniform Distribution***:
  - Let X be a random quantity taking values in $[0, 1]$.
  - Let I be a sub-interval of $[0, 1]$.
  - Denote the event $X \in I$ by $\{I\}$.
  - If for any two sub-intervals I_1 and I_2,
$$\{I_1\} \succcurlyeq \{I_2\} \Leftrightarrow \text{length } I_1 \geq \text{length } I_2$$
    then X has a uniform distribution on $[0, 1]$.

- **Theorem 3**: Under Axioms 1-5, for every $A \in S$, there exists a unique $x \in [0, 1]$, such that $A \sim \{[0,\ x]\}$, and we define $P(A) = x$.

- **Theorem 4**: As defined above P is the unique P which agrees with $\succcurlyeq$. Furthermore P satisfies the well known Kolmogorov conditions:

i. For every $A \in S$, $P(A) \geq 0$;

ii. $P(S) = 1$, and

iii. $P(\bigcup_1^\infty A_i) = \sum_1^\infty P(A_i)$ if $A_i \cap A_j = \emptyset, \forall i, j$.

- Discussion:
  - In de Finetti's development only finite additivity matters; countable additivity is an expedient (even to Kolmogorov).
  - Axioms 1-4 specify how $\succcurlyeq$ should behave.
  - Axiom 5 prescribes how to quantify uncertainty. It is a yardstick of quantification. It underlies all the work on the specification of explicit subjective probability

- I call Axiom 5 the ***axiom of acrobatics*** because in the context of a finite S it requires that the probability assessor fill in the sample space by an auxiliary imagined experiment, such as a spinning wheel of unit circumference, and that the experimenter compare the relative likelihood of any event of interest in S with that of any event of the form $\{I\}$ in the imagined experiment.

- Ramsey (1926) and Savage (1954) approach the simultaneous derivation of subjective probability and utility by imposing conditions on $\succcurlyeq *$, an assessor's ***preference relationship*** wherein $A \succcurlyeq * B \Rightarrow B$ is less preferred than A. That is, they consider an assessor's ***actions***.

- de Finetti (1974) also considers ***scoring rules*** and imposes requirements on these to arrive upon a subjective probability.

7. *Violation of Axioms:*
The Ellsberg Paradox

- Axioms 1-5 need to be satisfied to ensure that relative likelihood judgments can be represented by a unique probability distribution.

- For this we need to consider an infinite number of events and ensure an internal consistency vis-à-vis the relationship $\succcurlyeq$. A humanly impossible task.

- The axioms are therefore considered to be ***normative***; i.e. the ideal consistency that one should strive to achieve in one's judgments. In actuality, individuals may fail to achieve such consistency. Ellsberg (1961) produced a striking example.

- Because people violate the axioms of subjective probability, individuals have cast pallor on the axioms and as a consequence, probability itself.

- de Finetti and Savage respond by saying that such violations are a reflection of incoherent actions by individuals, not a poor reflection of a normative theory.
- But Humphreys rejects probability on grounds that it does not encapsulate causality, because probability unlike causality is symmetric.

Open Questions:

- Does nature itself subscribe to the axioms of coherent behavior that is expected of humans?
- In Feynman (1951), quantum theory denies the addition rule for electrons in the double slit expt.
- Preliminary work by Kevin Wilson and myself proves the great Feynman to be wrong.
- Physicists (Deutch & Wallace) have induced quantum probability via Bayesian decision theory.
- Subjective probability in quantum physics is a rewarding, but tough, frontier for us to consider!
- Remember Hilbert! He spent a year studying it.

Thanks

- For coming and listening to me.
- National Science Foundation's *DMS*, the *Defense Threat Reduction Agency*, and the Army Research Office's, *Network Science Division,* for spawning my interest in this topic and for sponsoring this effort initially; now supported by City University's start up grant.
- I intended to give this as a Plenary Talk at *Moscow State University* in a conference honoring Gnedenko's centenary in the same room that Kolmogorov lectured.
- But Shriyaev vetoed the idea in favor of filtering, forcing me to give a talk more boring than the one today. But that talk is now published in *Stat. Science*; this one is in limbo!